Dimensional Analysis

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In a mathematical world:

$$1 = 1$$

$$1 \neq 60$$

In a physical world:

$$1\ \mathrm{h} \neq 1\ \mathrm{km}$$

$$1\ \mathrm{h} = 60\ \mathrm{min}$$

What is a physical dimension?

"What is a dimension? It is simply a tag we attach to a quantity in an equation expressing some physical law, no more." McNish, 1957.

The only way to define a dimension is in terms of some physical law:

"One view held at present is that the definition of a physical quantity has been given when the procedures for measuring that quantity have been given. This view is called the operational view because the definition is, at root, a set of laboratory operations leading ultimately to a number with a unit." (Halliday and Resnick, 1963)

If dimensions are defined by physical laws, then there are a vast number of dimensions…
How to cope with all those dimensions?

By using a set of primary dimensions with 2 characteristics:
- any measurable variable can be assigned to one or a combination of primary dimensions,
- there should be either no or minimal redundancy between these primary dimensions.

The 7 primary dimensions of the International System of Units (SI) and their symbols (Thompson and Taylor, 2008):

- Length: L
- Mass: M
- Time: T
- Temperature: Θ
- Electric current: I
- Luminous intensity: J
- Amount of substance: N

The symbol for dimensionless variables is: 1

The dimension of any quantity Q can be written in the form of a dimensional product:

Dimension of $Q = L^{\alpha} M^{\beta} T^{\gamma} \Theta^{\delta} I^{\varepsilon} J^{\zeta} N^{\eta}$

where the exponents, which are generally small integers (positive, negative, or zero), are called the dimensional exponents

Units and physical dimensions are different!

A **unit** is a particular physical quantity, defined and adopted by convention, with which other particular quantities of the same kind are compared to express their value.

The **value of a physical quantity** is the quantitative expression of a particular physical quantity as the product of a number and a unit, the number being its numerical value. Thus, the numerical value of a particular physical quantity depends on the unit in which it is expressed.

There is a special relationship between dimensions and units. **A unit belongs to one and only one dimension**. While the dimension of a unit is unique, the units allowed to describe a dimension are not.

Definitions of the SI base units (Thompson and Taylor, 2008):

Dimension	**Unit**	**Standard**
Length (L)	meter (m)	The meter is the length of the path travelled by light in vacuum during a time interval of 1/299792458 of a second.
Mass (M)	kilogram (kg)	The kilogram is the unit of mass. It is equal to the mass of the international prototype of the kilogram.
Time (T)	second (s)	The second is the duration of 9192631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.
Electric current (I)	ampere (A)	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2 x 10^{-7} newton per meter of length.
Temperature (θ)	kelvin (K)	The kelvin is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.
Amount of substance (N)	mole (mol)	The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12
Luminous intensity (J)	candela (cd)	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540 x 1012 hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.

Dimensional analysis of an equation

[x] represents the physical dimension of x

$$x=y \Rightarrow [x]=[y] \quad \Rightarrow \quad [x]\neq[y] \Rightarrow x\neq y$$

$$\text{but } [x]=[y] \not\Rightarrow x=y$$

Example:

P=F/S where P is a pressure, F is a force and S is a surface

$[P]=[F]/[S]$ $\qquad$ $[F]=MLT^{-2}$ $\qquad$ $[S]=L^2$

$$\boxed{[P]= ML^{-1}T^{-2}}$$

The SI units for pressure are therefore: $kg.m^{-1}.s^{-2}$, $N.m^{-2}$, or Pa

Properties of the [ ] operator

[xy]= [x] [y]

[x/y]= [x]/[y]

$[x^n]=[x]^n$

[x]±[y]=[x]=[y]

[x/x]=1

$$\left[\frac{dx}{dy}\right]=\frac{[x]}{[y]}$$

$$\left[\frac{\partial^2 f}{\partial x \partial y}\right]=\frac{[f]}{[x][y]}$$

$$\left[\int_{x_1}^{x_2} f(x)\,dx\right]=\left[f(x)\right][x]$$

Exercises

1) Express the physical dimension of energy in terms of primary dimensions.

2) Can the following equation be correct?

$$\frac{E}{l} = \frac{a}{2\pi}\int_0^{+\infty} f(x)dx$$

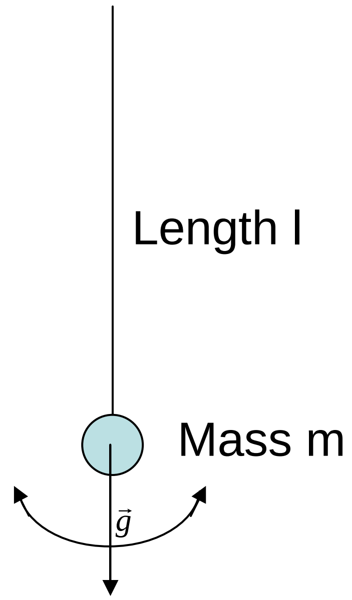
Where E is energy, l is length, f(x) is pressure, x is length, a is a dimensionless constant

Dimensional analysis can also help solve some problems when all equations are forgotten…

What is this pendulum's period?

$[T_{pendulum}]=[m]^{\alpha}[l]^{\beta}[g]^{\gamma}$

$[T_{pendulum}]=T$
$[m]=M$
$[l]=L$
$[g]=L.T^{-2}$



$$T = M^{\alpha}L^{\beta+\gamma}T^{-2\gamma} \quad \Longrightarrow \quad \begin{cases} \alpha = 0 \\ \beta = \frac{1}{2} \\ \gamma = -\frac{1}{2} \end{cases}$$

$$\boxed{T_{pendulum} \propto \sqrt{\frac{l}{g}} \quad \left(\text{the complete formulae is: } T_{pendulum} = 2\pi\sqrt{\frac{l}{g}}\right)}$$

A couple of take-home tips…

1) Always check that your equations are homogeneous.
2) Don't forget that the arguments of transcendantal functions **MUST** be dimensionless.
3) Always try to work with the International System of Units (SI).
4) The first letter of the symbol of a unit must be capitalised if it is named after a scientist (except for litre where both l and L are accepted).
5) Avoid numerical values in mathematical formulaes. Associated units do not figure in formulaes and therefore can lead to misleading dimensional analyses.
6) If you use the expression «dry weight», then use N or kg.m.s^{-2} to express the corresponding results (alternately, you can consider «dry biomasses», it's easier!).
7) Sum of temperatures are not homogeneous with temperatures!

References

- McNish AG. 1957. Dimensions, Units, and Standards. Physics Today. 1957. 19-25.

- Resnick R, Halliday D. 1987. Physics Part I, Forty Second reprints (Wiley Eastern Limited, New Delhi).

- Thompson A, Taylor BN. 2008. The International System of Units (SI). NIST Special Publication 811.